THE

MATHEMATICAL GAZETTE.

LONDON:
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

JOHN BOLYAI'S "SCIENCE ABSOLUTE OF SPACE."

II.

We have already explained pp. (25-31) that there are two types of uniform three-dimensional space, viz., finite or elliptic space, in which the geodesics or straight lines are closed and of constant length, and infinite or hyperbolic space, in which the geodesics are unclosed and of infinite length. Euclidean or parabolic space may be regarded as a special third type, or as the dividing case between the two general types; but it is best to regard it as a variety of hyperbolic space, owing to the reality of the infinite. In elliptic space all geodesics through a point either return without intersecting again, or have at most one other fixed point of intersection. In hyperbolic space it does not appear necessary that a geodesic should consist of only a single branch, though this is of course the simplest case; if the geodesics have several branches, all geodesics through a point will have a fixed point on each branch, and two geodesics which intersect will have as many points of intersection as they have branches. When we consider two-dimensional space we find that, even in the elliptic type, the geodesics may have any number of intersections, and may be unclosed and infinite. The proof to the contrary on p. 31 does not apply when the geometry is confined to a surface.

The distinctive property of uniform space, following at once from the principle of superposition, is that the sides and angles of any triangle are connected by three fixed independent relations. Directly connected with this is the remarkable theorem that the fixed relations (if any) which exist for any triangle in space can be determined, provided only it is known that there are three of them. They cannot, in fact, be any other than the well-known formulae of Euclidean spherical trigonometry, or their equivalents in hyperbolic functions. The former lead to elliptic and the latter to hyperbolic geometry. Hence it follows that there are no other types of uniform space than

the elliptic and hyperbolic, whatever varieties there may be of each type.

The analytical geometry of great circles on spheres with finite radii is the same in all kinds of uniform space, forming the most intimate connecting link between them. Small spheres have approximately the same shape in all space, whether uniform or not; but as the radii become greater they diverge more and more from the shape of Euclidean spheres. In uniform space the locus of all points at a given distance from a plane (geodesic surface) on both sides is a sphere, or else consists of two antipodal spheres which may be regarded as forming one complete sphere. A plane is therefore formed by the coincidence of a sphere with itself, back to back. Plane hyperbolic geometry is the same as that which would have place on a Euclidean sphere with purely imaginary radius; and holds in all essential respects, as shown by Beltrami, on any real Euclidean surface of constant negative curvature.

The properties of simple elliptic space are in some respects less easily grasped than those of other uniform spaces. Any plane encloses the whole of space, *i.e.* the whole of space is inside it, and it has no outside. If the plane be regarded as having infinitesimal thickness, the outside consists only of the material of which the plane is made. A line can pass from one side of the plane to the other without crossing it, as may be seen by following the course of any geodesic which does not lie in the plane. Also a geodesic which makes a single circuit in the plane ends on the other side from which it started; and the plane is not divided into two parts by cutting it along a geodesic. The special difficulty is in forming a mental picture of planes or any surfaces on which closed lines can be drawn with only one point of intersection. This property can be illustrated on Euclidean surfaces possessing singularities. If, for example, a cylinder be bent until the two arms lie in contact along a generating line, we get a surface of the kind described. Each of the closed lines must pass over the bent portion of the cylinder, and extend to both arms, passing from one on to the other by crossing the line of contact.

There is also a very interesting way of practically converting Euclidean into elliptic or hyperbolic space. This consists in assuming a new definition of distance. Suppose, for example, the distance between any two points P, Q in Euclidean space to be defined as $PQ \div OP \,.\, OQ$, where O is any fixed point; and let this distance be denoted by $\overline{PQ}$. Take any two fixed points, A, B, and consider the locus of a point P such that $\overline{AP}+\overline{PB}=\overline{AB}$, or $AP \,.\, OB+PB \,.\, OA=AB \,.\, OP$. The locus of P is not a surface, as would generally happen from a single condition, but a line, viz. the circle OAB. Also if P, Q, R be any three

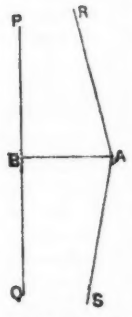
points on the circle OAB we have $\overline{PQ}+\overline{QR}=\overline{PR}$. These facts suffice to make the above definition of distance an appropriate one for some system of geometry, in which the geodesics or shortest lines are circles through O. Further, if we examine the relations connecting the sides and angles of any triangle ABC formed by such geodesics, we find that the sum of the angles is two right angles, and that the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ are proportional to the sines of the opposite angles. Hence it follows that the geometry which applies to these geodesics is Euclidean. The geodesic surfaces are spheres through O. To get the idea of rotation we have to imagine an infinite series of spheres drawn through the same fixed circle, each sphere becoming visible in succession while the rest disappear. Every property of Euclidean straight lines and planes will be true of circles and spheres through O, provided the distance between any two points is defined as above. All this is only equivalent to ordinary inversion; but the method can be extended. If for geodesics we take circles whose power with respect to a fixed point O is constant (instead of zero as above), there is a corresponding appropriate definition of distance, and the geodesic surfaces are in this case spheres whose power with respect to O is the same as that of the circles. The geometry which applies to these is hyperbolic or elliptic according as the power is positive or negative. If the power is positive the sphere with centre O and radius equal to the square root of the power is cut orthogonally by all the geodesic circles and spheres; and the extent of space to which the geometry applies is confined either to the interior or to the exterior of the orthogonal sphere. This is the only way, by merely taking a new definition of distance, of making elliptic and hyperbolic geometry applicable to Euclidean three-dimensional space; but the same thing can be effected on a Euclidean plane in an infinite number of ways.

In respect to uniform space Bolyai was right in assuming that his geometry was the "Science Absolute"; for all real curves and surfaces in elliptic and Euclidean space, and many imaginary ones, have real equivalents in hyperbolic space. In describing the properties of hyperbolic space, we shall, following Bolyai, speak of straight lines instead of geodesics, in the same sense that we have already used the term plane as equivalent to geodesic surface.

The assumptions made by Bolyai may be stated as follows: (i.) that space is uniform, so that proof by superposition is valid; (ii.) that straight lines are unclosed and infinite; (iii.) that two straight lines cannot enclose a space; (iv.) that a plane, defined as the surface generated by the rotation of a straight line about a perpendicular straight line, has the property that the straight line joining any two points on the plane lies wholly within it.

We have attempted to show (p. 31) that (iii.) and (iv.) follow from (i.) and (ii.); but, whether this is actually the case or not, it is evident that (iii.) and (iv.) must be classed as distinct assumptions so long as they are not proved. These assumptions being granted, it follows that the first twenty-eight propositions of Euclid Book I., and the first nineteen propositions of Book III., will also hold in Bolyai's geometry, since the same methods of proof will be applicable. Also some propositions of Book XI. will hold. Thus two planes which meet intersect in a straight line. A straight line PQ perpendicular to two straight lines PA, PB is normal to the plane PAB, *i.e.* is perpendicular to every straight line in the plane through P. Again, any two normals to a plane lie in one plane, although they are not parallel. This is used by Bolyai more than once, and may be proved as in Euclid XI. 6. Hence it follows that if two planes are at right angles at one point, *i.e.* if each contains the normal to the other at that point, they will be at right angles at every point in their line of intersection. The angle between two planes is taken to be the same as in Euclid, viz. the angle between the lines in the two planes perpendicular to their line of intersection at any point. It is assumed that this angle is the same at every point on the line of intersection; this is by no means evident, but may be proved from considerations of symmetry, and in other ways.

Bolyai defines a parallel in his first paragraph (§ 1) as follows: "If the line BP is not cut by AR, situated in the same plane, but is cut by every straight line within the angle BAR, AR is called parallel to BP." It is clear that there is only one parallel through A to BP, and that the angles PBA, RAB must be together equal to or less than two right angles; if equal, it is proved (§ 13) that the same property must hold for any pair of parallel lines crossed by any third line, and that consequently all the geometry of Euclid is applicable. Bolyai therefore, like Lobatchewsky, makes the contrary supposition, viz. that the sum of the angles PBA, RAB is less than two right angles, and proves (§ 14) that the same property must then hold for any pair of parallel lines. On this hypothesis it is to be noticed that the parallel AS to BQ or PB produced is not in a straight line with AR, but makes with it, on the side on which PBQ is, an angle less than two right angles; and that a straight line drawn in either of the external angles between AR and AS will not cut PQ at all. Thus some of the fundamental properties of Euclidean lines and curves have no place in the geometry of Bolyai. Parallel planes do not exist, since the parallels through a given point



to all lines on a plane do not themselves lie on a plane, but on a cone with the given point as vertex.

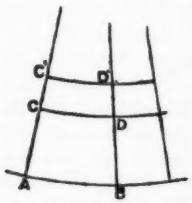
From Bolyai's second to his fifth paragraph (§§ 2–5) there is nothing which is not self-evident; these lead up to § 6, which proves that two lines which are parallel maintain their property of parallelism for the same direction of extension at all their points. In § 7 we have the interesting theorems, very different to the corresponding theorems in Euclid, that two lines which are parallel to the same line are parallel to one another, and two planes through two parallel lines intersect in a third line parallel to both. From this it may be assumed in theory that any system of parallel lines meet in the same point at infinity; and that a straight line meets infinity in opposite directions in two distinct points. § 9 contains a most important and remarkable theorem, viz., that two planes which cut a third plane along parallel lines must intersect if they make with it two angles on the same side which are together less than two right angles. It follows as an immediate consequence that the sum of the three interior angles of the three intersecting planes is equal to two right angles; but Bolyai reserves this in order to introduce it with greater effect later on (§ 21).

The first ten pages (§§ 1–15) make it abundantly clear that Euclidean is but a particular case of the more general geometry founded on the rejection of the parallel axiom. From this point Bolyai begins to develop the latter to its full extent. Starting with a plane, and any straight line AM on it, he defines the *L-line with axis* AM as the locus of all points B in the plane such that AM and the parallel through B to AM make equal angles with AB and BA respectively; and the *F-surface with axis* AM as the locus of all points B in space possessing a like property. These L-lines and F-surfaces do not exist in elliptic space, and become straight lines and planes in Euclidean space. Bolyai proves that an L-line has the properties of a circle, and an F-surface those of a sphere, their centres being at infinity. Every plane through AM cuts the F-surface with axis AM in an L-line with axis AM; and every plane through A oblique to AM cuts the F-surface in a finite circle. L is a uniform line, and F a uniform surface. Again, AM is a normal to the F-surface and the parallel to AM through any point B on the surface is both a normal and an axis of the F-surface, just as AM is. Finally follows a theorem (§ 21), which is undoubtedly the most remarkable property of hyperbolic space, that the sum of the angles of any triangle formed by L-lines on an F-surface is equal to two right angles. On this theorem Bolyai remarks: "From this it is evident that Euclid's twelfth Axiom and all things which are claimed in geometry and plane trigonometry hold good absolutely in F, L-lines taking the place of straight lines. Therefore the

trigonometric functions are defined here to have the same values as in Euclidean geometry; and the perimeter of the circle on F with L form radius r is $2\pi r$; and likewise the area of the same circle on F is πr^2; where $\pi = 3{\cdot}1415926\ldots$" The whole of Euclidean geometry is true for uniform curves no less than for straight lines, provided that, as is the case with L-lines on an F-surface, the assumptions made by Euclid hold with respect to them. In particular, two equiangular triangles need not be of the same size, but have their sides proportional; and on this property is based the definitions of the trigonometric functions. Bolyai, in fact, deduces his geometry of hyperbolic space from the Euclidean geometry of the F-surfaces. It should be noticed that a circle on an F-surface, like a circle on a sphere, is also a plane circle with a shorter radius.

From the last theorem two other fundamental theorems are deduced in §§ 25, 26, viz., that the sines of the angles of a rectilineal triangle are proportional to the perimeters of the plane circles described with the opposite sides as radii; and that the sines of the angles of a triangle formed by great circles on a sphere are proportional to the sines of the angles which the opposite sides subtend at the centre of the sphere. The trigonometry of great circles on finite spheres in any uniform space is the same as in Euclidean space.

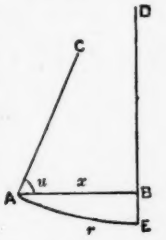
The proper place for the last-mentioned §§ 25, 26 would appear to be immediately after § 21; then should follow §§ 22-24, 28, 27, 29–31, in the order named. These eight paragraphs are in themselves very involved, and their difficulty is increased by the order in which they are given. § 22 proves that parallels to an L-line are also L-lines; *i.e.* if equal lengths AC, BD,... be taken on the normals to an L-line AB, the locus of the points C, D,... is an L-line having the same system of normals as AB. § 23 proves that the ratio of the arcs AB, CD is a function of the distance AC between the L-lines, and is independent of the length of the arc AB. If x denotes the length of AC, the ratio $AB:CD$, which is a function of x, is denoted by the corresponding capital letter X. It is proved in § 24 that $\log X : \log X' = x : x'$; in § 28, that $X = \sin ADB : \sin DAC$; and in § 30, that $X = e^x$.



In § 27 Bolyai considers the locus of a point whose perpendicular distance from a straight line AB is given. Any such line is a uniform curve, concave to AB, whose curvature is greater than that of a straight line, and less than that of an L-line. The complete locus on both sides of the line may be regarded as a circle without a centre, and may be called a hyperbolic circle, since it has two branches There are thus four kinds of uniform lines, viz., straight

lines, hyperbolic circles with two branches, L-lines or parabolic circles with infinite radii, and elliptic circles with finite radii.

In § 29 it is proved that if a parallel AC is drawn to a given line BD, at a distance $BA = x$ from BD, then the angle CAB is given by the formula $\cot \frac{1}{2}u = X$, or $\operatorname{cosec} u = \cosh x$ (by § 30). In § 30 an expression is sought for the perimeter of a plane circle in terms of the radius x. Let the L-line with axis AC cut BD in E; and let the arc AE be denoted by r. Then if $EACD$ be rotated round ED, the arc EA describes a portion of the F-surface having ED and AC for axes; and the point A describes a circle on the F-surface with centre E and radius r, and at the same time the plane circle with centre B and radius x. Hence the perimeter of the plane circle with radius x is $2\pi r$ (by § 21). It is shown in § 30 that the ratio $r : \cot u$ is a constant independent of r; denoting this constant by i, so that $r = i \cot u$, it is next shown that the corresponding ratio I (the ratio of L-lines at a distance i apart) is no other than the base e of natural logarithms, "which noted quantity shines forth here also." This constant length i serves as an absolute unit of measurement in the hyperbolic space considered; hence in what follows we take $i = 1$. By § 24, we have $\log X : \log I = x : i$, or $X = e^x$, since $I = e$, $i = 1$. Hence the perimeter of the plane circle with radius x is given by



$$2\pi r = 2\pi i \cot u = 2\pi \cot u = \pi(\cot \tfrac{1}{2}u - \tan \tfrac{1}{2}u)$$
$$= \pi(e^x - e^{-x}) = 2\pi \sinh x.$$

Also by § 29, since $\cot \frac{1}{2}u = X = e^x$, we have

$$\operatorname{cosec} u = \tfrac{1}{2}(\cot \tfrac{1}{2}u + \tan \tfrac{1}{2}u) = \cosh x.$$

Having thus found the perimeter of the circle, Bolyai obtains (§ 31) the trigonometric formulae for plane right-angled triangles, "whence the solution of all triangles is easy." The following formulae are proved, C being the right angle,

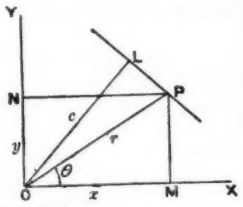
$$1 : \sin A = \sinh c : \sinh a, \quad \cos A : \sin B = \cosh a,$$
$$\cosh c = \cosh a \,.\, \cosh b = \cot A \,.\, \cot B.$$

These are the same as the formulae for Euclidean spherical right-angled triangles, except that hyperbolic functions of the sides appear in the place of trigonometric functions. A wrong construction is given for the proof of $\cosh c = \cosh a \,.\, \cosh b$; instead of drawing "$\beta\beta'$ and $\gamma\gamma'$ parallel to aa'," it should be "draw aa' perpendicular to $\beta a\gamma$, $\beta' a' \gamma'$ perpendicular to aa', and $\beta\beta'$, $\gamma\gamma'$ perpendicular to $\beta' a' \gamma'$."

The remainder of the book (§§ 32–43) shows "briefly the mode of solving problems" and making constructions. The area of the circle with radius x is found to be $4\pi \sinh^2\frac{x}{2}$; the volume of a sphere to be $\pi(\sinh 2x - 2x)$, and the area of a segment of a sphere to be equal to that of a plane circle whose radius is the chord of half the arc. The area of any triangle is shown to be measured by the defect of the sum of the angles from two right angles. Bolyai also says that a rectilineal figure can be drawn equal in area to a circle; but, in the proof he gives, it is far from clear whether he means a plane circle or a circle on an F-surface, or again a particular circle or any given circle; and, whatever he may mean, he fails on this point to be convincing.

In the matter of constructions Bolyai shows how to draw a parallel to a given line through a given point, and how to draw a parallel to a given line which is perpendicular to a second given line. His other constructions are not made in the plane, and are scarcely satisfactory. In order to make more than a very limited number of plane constructions, it would be necessary to have, in addition to the compasses and ruler, an instrument for drawing curves at any given distance from a given straight line.

Bolyai only makes some slight reference to analytical geometry. In this connection the following results may be of interest, as showing that the pure geometry of Bolyai leads directly to non-Euclidean analytical geometry.



Let OX, OY be rectangular axes of co-ordinates, P any point on a given straight line whose equation is required, OL perpendicular to the given line, and PM, PN perpendiculars to the axes. Then in the triangle POM we have

$$\cos POM = \tanh OM \,.\, \coth OP,$$

or $$\tanh OM = \tanh OP \,.\, \cos POM.$$

Hence putting

$$\tanh OM = x,\ \tanh ON = y,\ \tanh OP = r,\ \tanh OL = c,\ POX = \theta,\ LOX = \alpha,$$

and calling x, y the rectangular, and r, θ the polar coordinates of P, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ c = r\cos(\alpha - \theta), \quad \ldots\ldots(1)$$

$$\therefore\ x\cos\alpha + y\sin\alpha = c.$$

Hence, noticing that x, y, r, c are hyperbolic tangents, and therefore, *in reference to real points*, numerically less than unity, it follows that the equation to any real straight line may be written in the form

$$xx' + yy' = 1,\ \text{where}\ x'^2 + y'^2 > 1. \quad \ldots\ldots(2)$$

Also for any real point x, y, we have $x^2 + y^2 = r^2 < 1$; and the portion of the plane at infinity or the *absolute* is represented by the equation $x^2 + y^2 = 1$.

The equation to the circle with centre x', y', and radius γ, is

$$a(1 - x^2 - y^2) = (1 - xx' - yy')^2,\ \text{where}\ a = (1 - x'^2 - y'^2)\cosh^2\gamma \quad \ldots\ldots(3)$$

The equation to the hyperbolic circle at distance δ from $xx'+yy'=1$ is

$$a(1-x^2-y^2)=(1-xx'-yy')^2, \text{ where } a=(x'^2+y'^2-1)\sinh^2\delta. \quad \ldots\ldots(4)$$

These are also formulae giving the distance between the points x, y and x', y', and the distance of the point x, y from the straight line $xx'+yy'=1$. In taking the square root of (3) the positive sign should be prefixed, since all the factors as written are necessarily positive.

The equation to the L-line, or parabolic circle, with the point $\cos\alpha$, $\sin\alpha$ at infinity for centre, all lines through this point being normal axes, is

$$a(1-x^2-y^2)=(1-x\cos\alpha-y\sin\alpha)^2. \quad \ldots\ldots(5)$$

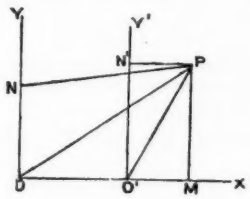
The condition that $xx'+yy'=1$, $xx''+yy''=1$ are perpendicular, is

$$x'x''+y'y''=1. \quad \ldots\ldots(6)$$

Transformation of equations for any change of rectangular axes can be effected without much difficulty. If the origin be unchanged and the axes be turned round an angle α, we have

$$x=r\cos(\theta'+\alpha)=x'\cos\alpha-y'\sin\alpha, \quad y=x'\sin\alpha+y'\cos\alpha. \quad \ldots\ldots(7)$$

The origin may be changed to any point in two steps, by first changing to a



point O' on OX, and then to a point on the new axis $O'Y'$. If $OO'=\gamma$, we have

$$x=\tanh OM=\tanh(OO'+O'M)=\frac{x'+\tanh\gamma}{1+x'\tanh\gamma}; \quad \ldots\ldots(8)$$

and $\tanh PM=\tan POM\,.\,\sinh OM=\dfrac{y}{x}\,.\,x\cosh OM=y\cosh OM=y'\cosh O'M,$

$$\therefore \quad y=\frac{y'\cosh O'M}{\cosh(OO'+O'M)}=\frac{y'\operatorname{sech}\gamma}{1+x'\tanh\gamma}. \quad \ldots\ldots(9)$$

These formulae show that the degree of an equation in x, y is not altered by any transformation of rectangular axes, as might be inferred from the fact that a straight line has always a linear equation. We may then define a conic as a curve whose equation is of the second degree. The equation of a real conic, or any real curve, must be such that real values x, y of the variables can satisfy the relation $x^2+y^2<1$. A straight line cannot cut a conic in more than two real points; and it may cut it in one real point without cutting it in a second.

Any real conic (with one special exception) has at least one real axis of symmetry, such that any chord perpendicular to the axis is bisected by the axis. One proof of this depends on the property of a cubic equation stated in Problem 119, p. 70. Also one axis of a conic cuts the curve in a real vertex A. If this axis cuts the conic in a second real vertex A', the line at right angles to AA' through its middle point C is also an axis; and every chord through C is bisected at C. Thus conics may be divided into central conics and non-central conics, or parabolas.

Central conics may be divided into ellipses, which cut the absolute $x^2+y^2=1$ in four imaginary points; and hyperbolas, which cut the absolute in four real points, and have four real asymptotes. Parabolas in general cut the absolute in two real and two imaginary points, and have two real asymptotes; but they include also all conics which have contact at a single point with the absolute. Finally, circles are conics which have double contact with the absolute. The analytical theory of conics is in many respects the same as in Euclidean geometry, the great difference being that the coordinates of a real point may not have unrestricted real values. The formulae for the straight line joining two given points, for the tangent at a given point on a conic, for the polar of a given point, and the pole of a given line, are of precisely the same form, and obtained in the same way, as in Euclidean geometry.

The equation of any central conic may be reduced to the form $\frac{x^2}{a}+\frac{y^2}{b}=1$, where a is greater than b. Taking $c^2=\frac{a-b}{1-b}$, and $e^2=\frac{a-b(1+a)}{a}$, the various kinds of conics may be described in greater detail as follows:

I. *Ellipses* $(0<b<a<1)$. Ellipses are closed curves, with real foci S, S' $(0, \pm c)$, and real vertices A, A' $(0, \pm\sqrt{a})$, such that, P being any point on the curve,

$$PS+PS'=AA'.$$

(i.) If $e^2>0$, the ellipse has real directrices $(cx=\pm a)$, possessing the property that if PM be perpendicular to the directrix XM $(cx=a)$, then

$$\sinh PS : \sinh PM=e;$$

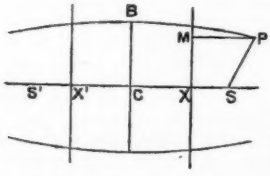
i.e. the perimeters of the circles with PS, PM as radii are in a constant ratio, less than unity.

(ii.) If $e'^2=-e^2>0$, the directrices are imaginary; but if S_1 is the point $0, \frac{c}{a}$, the relation in (i.) gives place to

$$\sinh PS : \cosh PS_1=e'.$$

The limiting case of an ellipse is given by $a=1$, when the vertices and foci are at infinity, and the ellipse becomes a hyperbolic circle.

II. *Convex Hyperbolas* $(0<b<1<a)$. The vertices and foci have disappeared in the imaginary region beyond infinity; and the curve has now two branches, one lying between the lines $y=\sqrt{b}$, $y=\sqrt{\frac{b(a-1)}{a-b}}$, and both lying between the lines $x=\pm\sqrt{\frac{a(1-b)}{a-b}}$.



(i.) If $e^2>0$, and S_1 be the point $0, \frac{c}{a}$, and PM_1 perpendicular to the line $cx=1$, then

$$\cosh PM_1 : \cosh PS_1=e.$$

(The points S, X, M in the figure should be S_1, X_1, M_1.) The limiting case is given by $a=\infty$, when the conic becomes the two straight lines $y=\pm\sqrt{b}$.

(ii.) If $e'^2=-e^2>0$, and PM, PM_1 perpendiculars to the lines $cx=a$, $cx=1$, then $\cosh PM_1 : \sinh PM=e'$.

A similar relation to this last, for two lines perpendicular to OY, holds for any convex hyperbola.

III. *Concave Hyperbolas* ($b<0<a<1$). The vertices, foci $(0, \pm c)$, and directrices ($cx=\pm a$) are all real; and we have

$$PS \sim PS' = AA', \quad \sinh PS : \sinh PM = e.$$

IV. *Conics with special properties.*

(i.) Conics for which $e^2=0$. The relation $\sinh PS : \sinh PS_1 = \text{constant}$ gives an ellipse for which $e=0$; $\cosh PM : \cosh PM_1 = \text{constant}$, a convex hyperbola; and $\sinh PS : \cosh PM = \text{constant}$, an ellipse or convex hyperbola. In all such conics the fixed points and lines S, S_1, XM, X_1M_1 may to some extent be chosen arbitrarily. In the ellipse the axis AA' subtends a right angle at any point on the curve; the directrices are at infinity; and S, S_1 are any pair of points $(0, c), (0, c')$ on the axis, such that $cc'=a$.

(ii.) Each of the relations

$$\cosh PS : \cosh PS_1 = \text{constant}, \quad \sinh PM : \sinh PM_1 = \text{constant},$$

$$\cosh PS : \sinh PM = \text{constant},$$

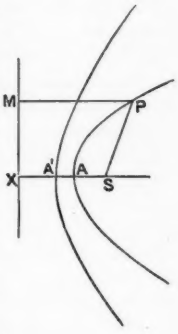
gives a pair of straight lines.

V. *Parabolas.* The equation of a parabola can in general be reduced to the form $\frac{x^2}{a}+\frac{y^2}{b}=x$, where a is positive and greater than 1. By changing the origin to $0, \frac{1}{a}$ the equation assumes the form $\frac{y^2}{b}=x+\frac{1}{a}$.

(i.) Parabolas in which $b<a$. This includes the case in which b is negative, the positive direction of the axis OX being then outwards. For parabolas of this kind there are two fixed points S, S_1 on the axis, and two fixed lines XM, X_1M_1 perpendicular to the axis, such that

$$\sinh PS : \sinh PM = \cosh PM_1 : \cosh PS_1 = \left(1-\frac{b}{a}\right)^{\frac{1}{2}}.$$

If we consider the conic with focus S, directrix XM, and vertex A on SX, such that $\sinh PS : \sinh PM = \sinh AS : \sinh AX$, it is clear that when A is near to S, the conic is an ellipse, and when near to X, a hyperbola. For a certain position of A the centre of the ellipse is at infinity; and if XA' be made equal to AS, the conic with vertex A' is a hyperbola with centre at infinity. For any position of the vertex between these extremes the conic is a parabola ($b<a$), which cannot in any sense be said to have a centre. If the vertex of the parabola bisects SX, its equation is $y^2=bx$.



(ii.) Parabolas in which $b>a$. In this case the relations in (i.) give place to

$$\sinh PS : \cosh PS_1 = \cosh PM_1 : \sinh PM = \left(\frac{b}{a}-1\right)^{\frac{1}{2}}.$$

[In both (i.) and (ii.), $\frac{x^2}{a}+\frac{y^2}{b}=x$ being the equation of the curve, and $(0, c), (0, c_1)$ the points S, S_1, and $dx=1, d_1x=1$ the lines XM, X_1M_1, then $c(c^2<1)$ and $d_1(d_1^2>1)$ are determined without ambiguity as the roots of $ab(1-x^2)=4x(a-x)$, and c_1, d as the roots of

$$a^2b(1-x^2)=4(b-a)(1-ax).]$$

Parabolas in which $b=a$ are included in IV. (i.).

(iii.) *Contact Parabolas.* The equation of a parabola which touches the absolute can in general be reduced to one of the forms $y^2=bx(1-x)$ or $y^2=b(x+1)$, but not to both forms. The only real vertex of the parabola $y^2=b(x+1)$ is at infinity.

The equation to the parabola which has contact of the second order with the absolute at the point 0, 1 can be reduced to the form $1-x^2-y^2=(1-x)y$, both axis and vertex being at infinity. This parabola is of fixed shape and size, and is remarkable as being the only conic without an axis of symmetry.

A parabola which has contact of the third order with the absolute is an *L*-line, or parabolic circle.

VI. Circles, or conics which have double contact with the absolute, include all uniform lines, as may be seen at once from formulae (3), (4), (5), p. 57.

Two non-intersecting straight lines are a limiting form of a convex hyperbola, two intersecting lines of a concave hyperbola, two parallel lines of a contact parabola, two coincident lines of a hyperbolic circle, and a single line of an ordinary parabola in which $b>a$.

EDITOR.

NOTES ON ELEMENTARY DYNAMICS. II.

IN problems where more than two bodies are simultaneously in collision, the rules usually stated in the text books are not, in general, sufficient for their solution. It has been suggested to me that I should continue the subject of Collision by a discussion of the method required in such cases.

In what follows I confine myself to the case of smooth spheres impinging on one another, or on fixed bodies, so that rotations do not fall to be considered.

I begin by explaining the method which seems to be at present in use, but which I have not seen explained in any text book; and thereafter I propose to discuss the limits of the validity of the method, taking up incidentally some other points connected with the subject, *e.g.*, its nomenclature.

The method in question is based on an assumed Law, which is a generalization of Newton's Experimental Law (*Principia*: Scholium on Cor. VI. to the Axiomata.)

Newton's Law is that the relative velocity of bodies, after directly impinging, bears a constant ratio to their relative velocity of approach. In the case of *oblique* impact, which is not referred to by Newton, it is usual to assume that the Law holds good for the components of the velocities normal to the surfaces of contact, the tangential components being unaltered.

Now this Law is equivalent (as is shewn in the text books) to the statement that the Impulse on either body *after* the instant of greatest compression (hereafter for brevity called

the G.C. instant) bears the same constant ratio to the corresponding Impulse *up to* that instant.

Now if *more than two* bodies are simultaneously in contact, it is clear that Newton's Law in its original form fails; for obviously if one body A is in collision with several bodies, B, C, ... simultaneously, the motion of A relative to B after impact depends not merely on *their* relative velocity before impact, but also on the motions of C....

But the Law, stated in terms of Impulse, can be generalized thus: *Assuming that the* G.C. *instant is the same for all the impacts between pairs of balls, then the value of* each *Impulse exerted by one ball on another* after *that instant bears a fixed ratio to its amount* up to *that instant, the value of the ratio for each pair being independent of the original velocities.* This *ratio* is the *coefficient of restitution*, and is usually denoted by e.

The method mentioned in the third paragraph above, is based on assuming this generalized Law.

Take first the case of *inelastic* bodies. Here at the assumed common G.C. instant the balls must be moving so that the resolved part of the relative velocity of each pair in contact normal to the surface of contact shall be zero. In the case of n bodies between which there are m simultaneous impacts, the motion being confined to one plane, there are $2n+m$ unknowns, each final velocity having two unknown components, and each impulse being unknown in *magnitude*. And there are $2n+m$ simultaneous linear equations to determine the unknowns, viz., two equations of motion for each body, of the form:

(resolved part of momentum *after*) − (ditto *before*)
= resolved part of impulse,

and m kinematic equations expressing the condition mentioned as to the normal velocities. If any body is fixed, then there are two unknowns the fewer, and two dynamical equations the fewer. Thus the problem of finding the impulsive actions and the resulting velocities is determinate, if the above assumptions be made. It is to be noted, however, that if any impulsive action comes out *negative* (which would mean pull instead of thrust) it implies that the supposed impact does not exist, and that the pair of bodies in question do *not* impinge on one another, but are driven asunder by the action of the others.

Next take the case of *elastic* bodies. The rule for solving questions of simultaneous impacts may be stated simply as follows: *Calculate the velocities and impulses as if the bodies were all inelastic, and we get the state of matters at the end of the "first part of the impact,"* i.e., *at the common* G.C. *instant. Then, by the assumed Law given above, write down the value of each impulse during the remaining part of the impacts, and*

write down the equations of momentum once more, using the values of the impulses thus found.

As a very simple example, take the case of three spheres of masses m_1, m_2, m_3 which have simultaneous direct impact in the same straight line, the velocities before impact being u_1, u_2, u_3, and the coefficients of restitution, e between m_1 and m_2, e' between m_2 and m_3.

In place of the two kinematic equations, we assume a single letter V to denote the common velocity at the G.C. instant. Denote by I, I' the amounts of impulse up to the same instant, I being that acting on m_2 from behind, and I' that on m_3 from behind. Then

$$-I = m_1(V-u_1) \quad \ldots\ldots(1)$$
$$I-I' = m_2(V-u_2) \quad \ldots\ldots(2)$$
$$I' = m_3(V-u_3) \quad \ldots\ldots(3)$$

These determine I, I', V. To get V, for instance, we add the three equations together, whence

$$(m_1+m_2+m_3)V = m_1u_1+m_2u_2+m_3u_3.$$

Denoting then by v_1, v_2, v_3, the final velocities, we have

Total impulse on m_1 $= -I(1+e)$ $\ldots\ldots(4)$

" " $m_2 = I(1+e)-I'(1+e')$ $\ldots\ldots(5)$

" " $m_3 = I'(1+e')$, $\ldots\ldots(6)$

Hence $m_1v_1 = m_1u_1 - I(1+e) = m_1u_1+m_1(v-u_1)(1+e)$,

$\therefore$ $v_1 = u_1+(V-u_1)(1+e) = V+e(V-u_1)$ $\ldots\ldots(7)$

Similarly, $v_3 = u_3+(V-u_3)(1+e') = V+e'(V-u_3)$ $\ldots\ldots(8)$

Again $m_2v_2 = m_2u_2+I(1+e)-I'(1+e') = m_2u_2+I-I'+eI-e'I'$

$= m_2u_2+m_2(V-u_2)+eI-e'I'$, by (2).

$\therefore$ $v_2 = V+eI/m_2-e'I'/m_2$ $\ldots\ldots(9)$

And in (9) we have only to substitute the values of V, I, I', determined by (1), (2), (3), in order to express v_2 in terms of the data.

It is worthy of note that the point of view explained in our previous note is also available to some extent here; only, in place of the motion of the common centre of gravity, we must now take the motions of the various bodies at the G.C. instant. In the above example, *e.g.*, equations (7) and (8) show that, so far as the bodies in the problem which have only *one* impact are concerned, we may use either of the equivalent rules:

rel. vel. after $= -e \times$ rel. vel. before,

vel. after = vel. before $-(1+e) \times$ (rel. vel. before),

vel. after = vel. at G.C. instant $-e \times$ rel. vel. before,

where the relative velocities in question are those with reference to the same body as it moves at the G.C. instant.

And in the particular case when all the e's are equal it is not

difficult to see that these rules would hold also for the bodies which undergo several simultaneous impacts (as m_2 in the above example).

These rules apply in the first instance to *direct* impacts, but if we understand the signs + and − to mean composition by the Parallelogram-law, and its reverse, they will apply also to oblique impacts. Or we may take them to refer to the resolved parts in a given direction.

So far we have simply stated and applied the usual assumptions made in dealing with problems of simultaneous impact. Before criticizing them, let us consider the very simple problem of three balls, A, B, C, each of unit mass, moving in the same straight line with velocities 12, 8, 3 respectively, the coefficient of restitution for each pair of bodies being $\frac{1}{2}$.

First suppose A to overtake B before C is overtaken. The velocities of A and B after impact will be $10-1$ and $10+1$, or 9 and 11 respectively. If then B overtakes C, their velocities will become $7-2$ and $7+2$, or 5 and 9 respectively. There will then be another impact of A and B, and their velocities thereafter will be 6 and 8. Thus the final velocities of A, B and C will be 6, 8, and 9.

Second, suppose B to overtake C before it is overtaken by A. There will be three impacts: of B on C, of A on B, and of B on C, and the state of motion at the various stages is shown thus:

	A	B	C
Initial vel.,	12	8	3
Vel. after B strikes C,	12	$4\frac{1}{2}$	$6\frac{3}{4}$
„ A „ B,	$6\frac{3}{16}$	$10\frac{1}{16}$	$6\frac{3}{4}$
„ B „ C,	$6\frac{3}{16}$	$7\frac{37}{64}$	$9\frac{15}{64}$

Thus the final velocities are different according to the order of the impacts. Now we might suppose the three successive impacts in each case to be so close together as to appear simultaneous. I think it is obvious from this that we might have an indefinite number of different results for the final velocities if we were to suppose the impacts interfering more or less with one another in different ways.

As a third supposition, take the case when the impacts of A and B, and of B and C are really simultaneous, *i.e.*, when the G.C. instant is absolutely the same for both, so that the three bodies would then be moving with a common velocity $=(12+8+3)\div 3=7\frac{2}{3}$. Assuming then the generalized form of the Law of Impact given above, and taking $e=\frac{1}{2}$, we at once find the final velocities of A, B, C to be $5\frac{1}{2}$, $7\frac{1}{2}$, 10.

It seems obvious by the foregoing example, and by general considerations, that some assumption of the sort above made is necessary if the problem is to be treated by elementary

methods; and further, that the experimental verification of the assumed Law would be much harder than in the case of a single impact, in as much as it would imply the power to observe whether the G.C. instants were really the same for all the impacts.

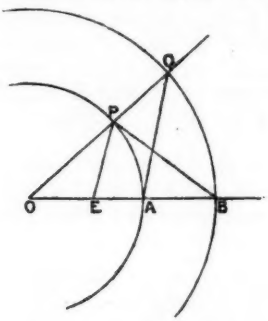
R. F. MUIRHEAD.

[*To be continued.*]

MATHEMATICAL NOTES.

30. *Proof of a well-known theorem in Geometry.*

Let O, A, B be three given fixed points in a straight line. From centre O describe two circles with radii OA, OB respectively. Through O draw *any*



straight line OPQ meeting the circumferences in P, Q; join QA, PB. Since $OP=OA$, and $OQ=OB$, therefore $PB=QA$ (Euclid I. 4). Draw PE parallel to QA meeting OA in E.

Then $OE : OA = OP : OQ = OA : OB$, hence E is a fixed point.

Also $PE : PB = PE : QA = OA : OB$. Thus we infer that the locus of a point P which moves so that its distances from two fixed points E, B are in a constant ratio is a circle. ANON.

31. *Elementary proof that the volume of a pyramid or cone is $\frac{1}{3}$ base × height, by assuming the properties of similar figures.*

Let the height of the cone be h; cut off a thin slice by a plane parallel to the base, and let the height of the smaller cone be h'.

The volumes are μh^3 and $\mu h'^3$, μ being some constant, and their difference

$$= \mu(h^3 - h'^3).$$

The volume of the slice cut off = base of cone $\times (h - h') = \mu(h^3 - h'^3)$;

$$\therefore \quad \frac{\mu(h^3 - h'^3)}{h - h'} = \text{base of cone},$$

i.e. $\quad \mu(h^2 + hh' + h'^2) = \text{base of cone},$

i.e. $\quad 3\mu h^2 = \text{base of cone in the limit};$

whence the volume $\quad \mu h^3 = \frac{1}{3}h \times \text{base}.$ PROF. A. LODGE.

EXAMINATION QUESTIONS AND PROBLEMS.

121. Find the general solutions of the equations:

$$\text{(i.)}\ \sin\left(\frac{\pi}{3}-\theta\right)+\sec\left(\frac{\pi}{6}+\theta\right)=2;$$

$$\text{(ii.)}\ \tan\theta+\tan 2\theta+\tan 3\theta=0. \qquad \text{(Edinburgh.)}$$

122. Having given that xyz is not zero, and that

$$yz+ay+bz=zx+az+bx=xy+ax+by=0,$$
$$yz+zx+xy=xyz,$$

prove that $a+b+3=0$. (Edinburgh.)

123. If x is positive, prove that $x+x^2+\ldots+x^n$ always lies between $\frac{1}{2}\{(n+1)x^n+(n-1)\}$ and $\frac{1}{2}n\{(n+1)x-(n-1)\}$. (Edinburgh.)

124. The bisector of the vertical angle of a triangle is not greater than the median from the same vertex. (Edinburgh.)

125. Show how to make a square section of a given regular tetrahedron. (Edinburgh.)

126. Find the minimum rhombus that can be cut out of a given parallelogram. (Edinburgh.)

127. Solve the equation $(x^2+bc)^2=4ax(x+b)(x+c)$.

128. If $ABCD$ is a parallelogram, X any point on BC, and Y any point on AX, prove that the triangles DXY, YBC are equal.

129. There is a half circle as ABC, whose diameter is AB, upon which is made another lesser half circle whose diameter is EA, so that the greater half circle being divided into two equal parts with the line DC, doth pass through the two circles in the points F, C, so that the part CF is 6 and the part BE is 9. How much is the diameter of each of these two circles?

(Captain Rudd's *Practical Geometry*, London, 1650.)

130. A shot of mass m is discharged from a gun which, together with the gun-carriage, is of mass M. The gun-carriage can slide on a smooth horizontal plane. Prove that for a given charge of powder the range is a maximum for an elevation $\frac{1}{2}\cos^{-1}\frac{m}{2M+m}$ of the gun. (Cambridge.)

131. Prove geometrically that

$$\sin 3A=4\sin A\,.\,\sin(60^\circ-A)\,.\,\sin(60^\circ+A).$$

(Cambridge.)

132. Prove that $\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}=21;$

$$\tan^4\frac{\pi}{7}+\tan^4\frac{2\pi}{7}+\tan^4\frac{3\pi}{7}=371.$$ (Cambridge.)

133. If P, P' are any two points on a central conic, and the normals at P, P' meet the major axis in G, G' respectively, prove that the projections of PG, $P'G'$ on PP' are equal. (Cambridge.)

134. Two men A, B, each of mass m, sit in loops at the ends of a rope passing over a smooth pulley, A being h feet above B. A ball, mass $\frac{m}{10}$, is given to B which he instantly throws to A, so that it just reaches him. Show that by the time A has caught the ball he has moved up a distance $\frac{2h}{19}$, neglecting the mass of the rope; and that A will cease ascending when he has risen a height $\frac{59h}{361}$ altogether. (Cambridge.)

135. If two chords PQ, RS of a circle cut a given diameter in points equidistant from the centre, then so also do PR, QS and PS, QR.

(Solution of this question by elementary geometry is desired.)

136. The inverse of a circle with respect to a point outside its plane is a circle.

137. Prove that in an acute-angled triangle R^2 lies between $\frac{1}{8}(a^2+b^2+c^2)$ and $\frac{1}{9}(a^2+b^2+c^2)$, and that in an obtuse-angled triangle R^2 is greater than $\frac{1}{8}(a^2+b^2+c^2)$.

138. Find the five linear factors of the expression

$$x^5+y^5+z^5-5xyz(x^2-yz).$$

139. Find the conditions that the cubic $ax^3+3bx^2+3cx+d=0$ should have three real roots between ±1.

140. If m, n, x are positive integers, of which m, n are prime to one another, prove that

$$1+x+x^2+\ldots+x^{m-1} \text{ and } 1+x+x^2+\ldots+x^{n-1}$$

are arithmetically prime to one another.

141. Prove that if the areas of the four faces of a tetrahedron are equal, each edge is equal to the opposite edge.

142. Two variable lines through fixed points make angles θ, ϕ with any two fixed directions respectively; trace the curve described by their point of intersection when

$$e^{a\theta}\sin\theta=e^{a\phi}\sin\phi.$$

143. Prove that the result of eliminating the constants in the general equation of a conic by differentiation is

$$\frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=0.$$

(Oxford.)

144. Supposing packs of cards to be arranged in all the possible $52!$ ways, each pack taking up one cubic inch of space, calculate approximately or exactly the size of a cubical box which would hold all the lot; and the length of time a ray of light would take to travel from one corner to the farthest opposite. R. P. ROYSTON.

145. Show that the feet of the perpendiculars drawn from the angular point A to the bisectors (internal and external) of the angles B, C of the triangle ABC lie on the join of the mid-points of AB, AC. G. RICHARDSON.

146. Prove geometrically that $\tan^{-1}13-\cos^{-1}\frac{7}{\sqrt{85}}=\frac{\pi}{4}$.

T. ROACH.

147. Show that

$$\cos^2(\theta+\gamma)+\cos^2(\theta-\gamma)-2\cos(\theta+\gamma).\cos(\theta-\gamma).\cos 2\gamma$$

is independent of θ; and give a geometrical interpretation.

E. M. LANGLEY.

148. If points on the surface of a sphere are represented by their orthogonal projections on the plane of the paper, prove that parallel circles on the sphere are represented by a set of similar ellipses whose foci lie on a circle concentric with the sphere, and passing through the points representing the poles of the circles.

A. LODGE.

149. In the process of converting 1/100103 into a repeating decimal the 273rd remainder is 100067, the 6126th remainder is 14, and the 7587th remainder is 100091. Find where the remainders 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 will occur. C. N. MURTON.

150. If x, y, z be the coordinates of any point referred to rectangular axes, and r, r' the distances of x, y, z from any two fixed points, prove that the function $V\equiv a\log(br/r')$ satisfies the partial differential equations

$$\frac{d}{dx}\left(\frac{1}{U}\frac{dV}{dx}\right)=\frac{d}{dy}\left(\frac{1}{U}\frac{dV}{dy}\right)=\frac{d}{dz}\left(\frac{1}{U}\frac{dV}{dz}\right)=\frac{1}{U}\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right),$$

where

$$U\equiv\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2;$$

and that there is no solution more general. EDITOR.

SOLUTIONS.

Solutions without solvers' names attached are by S. H. PLUMMER *or the* EDITOR.

Professor A. LODGE writes as follows on the solution of Problem 84, p. 46: The explanation of the fallacy in No. 84 does not seem to me to be the correct one. $\frac{x^2-y^2}{x-y}$ is always equal to $x+y$, and must therefore be equal to it even in the limiting case when $x=y$, provided the problem is such as to allow us to make x gradually equal to y. But the deduction that, because $\frac{x^2}{x}=\frac{y^2}{y}$ when $x=y$, therefore each equals $\frac{x^2-y^2}{x-y}$ is erroneous as can be seen at once if we apply the method of limits. The equation $\frac{x^2}{x}=\frac{y^2}{y}$ is not a true proportion, since these fractions are necessarily unequal when x is not equal to y, however nearly equal they may be, and the difference of the numerators divided by the difference of the denominators will not have the same limiting value as the fractions themselves. The *third* link in the chain is therefore the weak one.

Of course, if x and y are only intended to be different names for the same thing, so that $x=y$ is an *absolute* identity, the above reasoning does not apply. But in that case the equation should be $\frac{x^2}{x}=\frac{x^2}{x}=\frac{x^2-x^2}{x-x}$ which has any value we please, though its natural value is x, as can be seen at once by performing the division. If we bar ourselves from using the method of limits, we cannot of course deduce anything from a fraction of the form $\frac{0}{0}$; it is, as stated in the previous solution of this question, absolutely indeterminate, or as I should prefer to put it, absolutely meaningless.

(The wording of Problem 84 leaves it a perfectly open question as to whether x, y should be regarded as absolutely equal or as having an infinitely small difference. The former view is the one adopted in the solution on p. 46. Prof. Lodge advocates the latter view, and points out that the explanation is then incorrect. The solution as given illustrates an important principle, which cannot be evaded or disregarded, that a fraction which takes the form $0\div0$ is indeterminate in the absolute limit, no matter how the limit may be approached.—ED.)

Want of space prevents us from publishing several criticisms on the solution of Problem 71, p. 45. The various answers $\frac{1}{49}$, $\frac{1}{55}$, $\frac{1}{58\frac{3}{4}}$, and $\frac{1}{48\frac{1}{3}}$ approximately, have been given.

91. *Sum the series*

$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\dots \text{ to } n \text{ terms.}$$

By adding $\frac{1}{1-x}$ to the series we get

$$\frac{2}{1-x^2}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\dots=\frac{4}{1-x^4}+\frac{4}{1+x^4}+\dots=\dots$$

$$=\frac{2^{n-1}}{1-x^{2n-1}}+\frac{2^{n-1}}{1+x^{2n-1}}=\frac{2^n}{1-x^{2n}}.$$

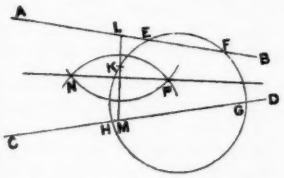
Hence the sum is

$$\frac{2^n}{1-x^{2n}}-\frac{1}{1-x}.$$

94. *Given two straight lines which are not parallel but do not meet on the sheet of paper on which they are drawn, make a construction on the paper for the bisector of the angle between them.*

Solution by R. F. MUIRHEAD.

Given AB and CD. Describe any circle $EFGH$ cutting them. With centre H and radius equal to GF draw an arc cutting the circle in K. Draw the arc KL with centre F. Draw LKM. The "axis" of LM, got by joining the intersections N, P, of two equal circles whose centres are L, M, is the required bisector.



This solution is probably the simplest possible by means of ruler and compasses if we reckon the simplicity according to the method of Geometrography as devised by Mr. E. Lemoine of Paris. He measures the "simplicité," or what in English idiom we should call the "complexity," of a construction by a number which is got by counting one for each time we adjust a compass point to a given point or line, and for each time we adjust a ruler to a given point, also for each straight line or arc drawn after these adjustments have been made. This gives of course only a rough, though a perfectly definite, estimate of the real "complexity" which might more properly be measured by the average time required by an expert draughtsman. The "complexity" of our construction is 18.

For a full account of "Geometrography" the reader is referred to Mr. Lemoine's papers in the *Reports* of the French Association for the Advancement of the Sciences for 1888 and 1892; also to a paper by Dr. J. S. Mackay in Vol. XII. of the *Proceedings* of the Edinburgh Mathematical Society.

95. *Resolve* $\Sigma(a^3b^3+c^3d^3)(a-b)(c-d)$ *into factors.*

Put $bc+ad=l$, $ca+bd=m$, $ab+cd=n$, when we get

$$\Sigma(l^3-3abcdl)(n-m)=\Sigma l^3(n-m)=(m-n)(n-l)(l-m)(l+m+n)$$
$$=-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)(bc+ca+ab+ad+bd+cd).$$

103. *Sum the series*

$$\frac{\cos 2x}{\sin 3x}+\frac{\cos 6x}{\sin 9x}+\frac{\cos 18x}{\sin 27x}+\ldots \text{ to } n \text{ terms.}$$

We have $$\frac{\cos 2x}{\sin 3x}=\frac{\sin x\,.\cos 2x}{\sin x\,.\sin 3x}=\tfrac{1}{2}\left(\frac{1}{\sin x}-\frac{1}{\sin 3x}\right).$$

Similarly $$\frac{\cos 2\,.\,3^{n-1}x}{\sin 3^n x}=\tfrac{1}{2}\left(\frac{1}{\sin 3^{n-1}x}-\frac{1}{\sin 3^n x}\right).$$

Hence the sum is $$\tfrac{1}{2}\left(\frac{1}{\sin x}-\frac{1}{\sin 3^n x}\right).$$

104. *Having given a directrix of a conic and the eccentricity and a point on the curve, show that the conic always touches a fixed conic having the same eccentricity.*

Let H be the fixed point, S the focus, and e the eccentricity of any one of the conics; let HS cut the conic again in P; draw HL, SX, PM perpendicular to the fixed directrix; and produce PM to N, making MN equal to HL. Then the parallel through N to LM is a fixed line.

Then $e=SH:HL=SP:PM=SH+SP:HL+PM=HP:PN$.

Hence P lies on a fixed conic with focus H and eccentricity e; and the conic with focus S touches the fixed conic with focus H at P, since they have the same eccentricity; which proves the theorem.

105. *Find the locus of the circumcentre of a triangle whose sides are the asymptotes and any tangent to a given hyperbola.*

Let TT' be any tangent touching the hyperbola at P, and meeting the asymptotes in T, T'; let O be the circumcentre of CTT', and CD, CD' radii of the conjugate hyperbola parallel to TT' and perpendicular to CO respectively, on opposite sides of the transverse axis. Then the angle $D'CT'$ is equal to CTT' or DCT; hence $CD'=CD=PT$. Also OPT is a right angle, and POT equal to TCT'. Hence $OT : PT$, or $CO : CD'$, is constant. Hence since OCD' is a right angle, the locus of O is a hyperbola similar to the conjugate of the given hyperbola.

107. *If two parabolas have a common focus, prove that they have only one real common finite tangent, and construct it.*

Let S be the common focus, and Z the point in which the tangents at the vertices intersect. Then supposing that any finite common tangent cuts the tangents at the vertices in P and Q, SP and SQ must both be perpendicular to PQ. This can only happen when P, Q coincide with Z. Hence the only common tangent is the line through Z perpendicular to SZ.

117. *Find the last* 13 *digits* in 52!. R. P. Royston.

Solution by the Proposer.

52! resolved into primes in the usual way is

$$2^{49}\,.\,3^{23}\,.\,5^{12}\,.\,7^{8}\,.\,11^{4}\,.\,13^{4}\,.\,17^{3}\,.\,19^{2}\,.\,23^{2}\,.\,29\,.\,31\,.\,37\,.\,41\,.\,43\,.\,47.$$

Couple the 12 fives with the same number of twos, giving the last 12 digits as zeros. Omitting factors which end in 1, we have only to look for the last digit in

$$2^{37}\,.\,3^{23}\,.\,7^{8}\,.\,3^{4}\,.\,7^{3}\,.\,3^{4}\,.\,3^{2}\,.\,3^{2}\,.\,7\,.\,3\,.\,7 \text{ or in } 2^{37}\,.\,3^{36}\,.\,7^{13}.$$

Again, 2^5 ends in 2, 3^4 in 1, and 7^4 in 1. Hence the last digit in $2^{37}\,.\,3^{36}\,.\,7^{13}$ is the same as in $2\,.\,7$ which is 4; so that the result is 4 followed by 12 cyphers.

118. *Prove that any triangle is acute-angled or obtuse-angled according as the semi-perimeter is greater or less than* $2R+r$. R. F. Davis

We have $$r=4R\sin\frac{A}{2}\,.\,\sin\frac{B}{2}\,.\,\sin\frac{C}{2}=R(-1+\Sigma\cos A),$$

$$2R+r=R(1+\Sigma\cos A).$$

Also $$s=R\Sigma\sin A=4R\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\,\cos\frac{C}{2},$$

$$s^2=2R^2(1+\cos A)(1+\cos B)(1+\cos C).$$

Hence $$s^2-(2R+r)^2=R^2\{1-\Sigma\cos^2A+2\cos A\,.\,\cos B\,.\,\cos C\}$$
$$=4R^2\cos A\,.\,\cos B\,.\,\cos C.$$

Therefore according as s is greater or less than $2R+r$, $\cos A\,.\,\cos B\,.\,\cos C$ is positive or negative, and the triangle is acute-angled or obtuse-angled.

119. *If the cubic equation* $x+\dfrac{a^2}{x+b}+\dfrac{c^2}{x+d}+e=0$ *has three real roots, one of the values of* $\dfrac{a^2}{(x+b)^2}+\dfrac{c^2}{(x+d)^2}$ *lies between* 0 *and* 1, *and the other two are both greater than* 1; *and if the cubic has only one real root,* $\dfrac{a^2}{(x+b)^2}+\dfrac{c^2}{(x+d)^2}$ *is greater than* 1. Editor.

Put $$u\equiv x+\frac{a^2}{x+b}+\frac{c^2}{x+d}+e,$$

then $$\frac{du}{dx}\equiv 1-\frac{a^2}{(x+b)^2}-\frac{c^2}{(x+d)^2}$$

Assume b to be greater than d, and suppose x to increase continuously from $-\infty$ to $+\infty$; then u passes through an infinite value four times, viz., when $x=-\infty$, $-b$, $-d$, $+\infty$. Also so long as u is finite $\frac{du}{dx}$ will be positive when u is increasing, and negative when u is decreasing. Further, when

x increases from	$-\infty$ to $-b$	$-b$ to $-d$	$-d$ to $+\infty$
u changes from	$-\infty$ to $-\infty$	$+\infty$ to $-\infty$	$+\infty$ to $+\infty$

Thus $u=0$ has at least one real root between $-b$ and $-d$.

I. Suppose $u=0$ has three real roots, viz., α, β, γ in ascending order. Then one of the three following cases must happen.

(i.) α, β lie between $-\infty$ and $-b$, and γ lies between $-b$ and $-d$. In this case when $x=\alpha$, u is increasing and $\frac{du}{dx}$ is positive; when $x=\beta$, u is decreasing and $\frac{du}{dx}$ is negative; when $x=\gamma$, u is decreasing and $\frac{du}{dx}$ is negative.

(ii.) α, β, γ all lie between $-b$ and $-d$. In the same way as in (i.), putting $x=\alpha$, β, γ, $\frac{du}{dx}$ is seen to be negative, positive, and negative respectively.

(iii.) α lies between $-b$ and $-d$, and β, γ lie between $-d$ and $+\infty$. When $x=\alpha$, β, γ $\frac{du}{dx}$ is negative, negative, and positive respectively.

In all three cases two values of $\frac{du}{dx}$ are negative and one positive. This proves the first part of the question, remembering that $\frac{a^2}{(x+b)^2}+\frac{c^2}{(x+d)^2}$ is necessarily positive.

If $u=0$ has two equal roots, the positive value of $\frac{du}{dx}$ and one negative value become zero, and the third value is negative.

II. Suppose $u=0$ has only one real root, viz., α. Then α must lie between $-b$ and $-d$; and when $x=\alpha$, u is decreasing and $\frac{du}{dx}$ is negative. This proves the second part of the question.

The Proposer would be glad to have a purely algebraic solution of the problem.

120. *Find the prime factors of*

5, 679, 431, 432, 056, 743, 205, 685, 679, 432. E. M. Langley.

Solution by J. Blaikie.

$$\begin{aligned}\text{The given number} &= 5679431,4320568(10^{14}-1)\\ &= 5679432(10^7-1)(10^{14}-1)=568(10^4-1)(10^7-1)(10^{14}-1)\\ &= 568(10^2-1)(10^2+1)(10^7+1)(10^7-1)^2.\end{aligned}$$

Also $568=2^3\,.\,71$, $10^2-1=3^2\,.\,11$, $10^7+1=11\,.\,909091$, $10^7-1=3^2\,.\,239\,.\,4649$.

The prime factors are $2^3\,.\,3^6\,.\,11^2\,.\,71\,.\,101\,.\,239^2\,.\,4649^2\,.\,909091$.

REVIEWS AND NOTICES.

Il passato ed il presente delle principali teorie geometriche. By Gino Loria, Professor of Geometry at the University of Genoa. (Carlo Clausen, Turin.) Professor Loria's historical treatise was first published in 1887 in Vol. 38, Series II., of the Memoirs of the Royal Academy of Science at Turin; and was translated into German in the following year by Herr Fritz Schütte. In its

present book form it has increased to about three times its original size, and, in addition to being an exhaustive history of Geometry, is an invaluable work of reference to the innumerable original memoirs on the subject scattered throughout the scientific journals of all countries. It will long stand unrivalled as an effectual aid for the advancement of geometrical research. Every branch of Geometry receives its due share of attention: the Geometry of the Ancients, the far-reaching productive theories of Chasles, Abel, Gauss, Riemann, Plücker, Möbius, Steiner, Staudt, Grassmann, Hamilton, Cayley, Cremona, etc., the geometry of general and special curves and surfaces, systems of curves, curves with double curvature, the form of geometric figures, singularities, generation and construction, configurations and congruences, correspondence and transformation, enumerative geometry, non-Euclidean geometry, and geometry of space with several dimensions. There is an index of nearly a thousand names referred to in the volume, but no index of subjects. The descriptive part of the work, as the title perhaps suggests, is more devoted to the exposition of theories than the enunciation of general propositions. Although the latter aspect is not neglected we cannot but think that if the learned author had dealt with it more fully the value and interest of the work would have been still further increased.

History of Modern Mathematics. By D. E. Smith, Ph.D., Professor of Mathematics in Michigan State Normal School. (Chapman & Hall.) This is a very useful condensed record of modern mathematics, not the first published in America, as witness Professor Cajori's *History of Mathematics*, and yet not the less to be welcomed, for there is ample need of more than one such record. Professor Smith's work covers little more than sixty octavo pages, being a reprint of the last chapter in the text-book on *Higher Mathematics*, edited by Professors Merriman and Woodward. Although probably compiled with the special object of illustrating and crowning the body of the text which precedes it, it forms by itself a work of considerable interest and value; and we hope that the author will sometime re-issue it in a much expanded form. It is divided into eighteen sections, with separate titles, including Theory of Numbers, Quaternions and Ausdehnungslehre, Substitutions and Groups, Infinite Series, Theory of Functions, and Analytical, Modern, and Non-Euclidean Geometry.

La risoluzione dei problemi numerici e geometrici. By Professor Bettazzi. This is a short account, with worked-out examples, of the different algebraical and geometrical methods which are used for the solution of problems. The greater part of it is naturally given up to the consideration of geometrical problems, and there is an appendix on the proof of theorems. The result is that a student has before him, in an orderly arrangement, a number of methods and general ideas with which he ought to be acquainted, but which he has only come across before scattered in a disconnected fashion in different text-books.

Riders in Euclid. By J. Hamblin Smith, M.A. (Longmans.) We should have assumed, had we not the evidence of this collection of riders before us, that all learners of Euclid would find a sufficiency of exercises in their text-books. The collection contains 786 examples, judiciously arranged and graduated, and is not complicated by any annoying digressions on methods of solution.

BOOKS AND MAGAZINES RECEIVED.

The American Mathematical Monthly. June-July.

The Mathematical Magazine. Vol. II., Nos. 9, 10.

Periodico di Matematica. May-June.

Journal de Mathématiques Elémentaires. June to October.

Proceedings of the Edinburgh Mathematical Society. Vol. XIV.

The Editor has on hand several *Mathematical Notes* which will be published in the next (February) Number. Notes not previously acknowledged have been received from E. T. Dixon, E. Fenwick, J. H. Herschkowitz, and E. Budden.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.